[10-01-14-T11]

Roots in complex numbers

[EX1] Solve in $\mathbb{C}$, $x^3 + 8\,i = 0$.

This is equivalent to finding all the 3rd roots of $-8\,i$.

First write $-8\,i$ in polar form $r[\cos\theta + i\sin\theta]$. $r = |-8\,i| = 8$, $\theta = \frac{3\pi}{2}$.

So, $-8\,i = 8[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}]$.

Then use the theorem $w_k = r^{\frac{1}{n}}[\cos(\frac{\theta+2k\pi}{n}) + i\sin(\frac{\theta+2k\pi}{n})]$, $k = 0, 1, 2, 3, \cdots, n-1$ and n is the number of roots.

$$w_0 = 8^{\frac{1}{3}}\left[\cos\frac{\frac{3\pi}{2}+0(2\pi)}{3} + i\sin\frac{\frac{3\pi}{2}+0(2\pi)}{3}\right] = 2[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}] = i\,2\sin\frac{\pi}{2} = 2\,i$$

$$w_1 = 8^{\frac{1}{3}}\left[\cos\frac{\frac{3\pi}{2}+2\pi}{3} + i\sin\frac{\frac{3\pi}{2}+2\pi}{3}\right] = 2[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}] = 2\left[\frac{-\sqrt{3}}{2} - i\,\frac{1}{2}\right] = \left(-\sqrt{3} - i\right)$$

$$w_2 = 8^{\frac{1}{3}}\left[\cos\frac{\frac{3\pi}{2}+4\pi}{3} + i\sin\frac{\frac{3\pi}{2}+4\pi}{3}\right] = 2[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}] = 2\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\,i\right] = \left(\sqrt{3} - i\right)$$

$\therefore\ x = 2\,i,\ -\sqrt{3} - i,\ \sqrt{3} - i$

You might find the product of these roots; it should be $-8\,i$. Let's see.

$2\,i\left(-\sqrt{3} - i\right)\left(\sqrt{3} - i\right) = -8\,i$

[EX2] Solve in $\mathbb{C}$, $x^3 - 1 = 0$.

This is equivalent to finding all the 3rd roots of 1.

First write 1 in polar form $r[\cos\theta + i\sin\theta]$. $r = |1| = 1$, $\theta = 0$.

So, $1 = 1[\cos 0 + i\sin 0]$.

$w_0 = 1\left[\cos\frac{0+0(2\pi)}{3} + i\sin\frac{0+0(2\pi)}{3}\right] = [\cos 0 + i\sin 0] = 1$

$w_1 = 1[\cos\frac{0+2\pi}{3} + i\sin\frac{0+2\pi}{3}] = [\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}] = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$w_2 = 1[\cos\frac{0+4\pi}{3} + i\sin\frac{0+4\pi}{3}] = [\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}] = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$\therefore\ x = 1,\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

You might find the product of these roots; it should be 1. Let's see.

$1\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1$